Pdf-01, H-1049, (1) Dr. Satish Kumar Gupta

Q.1. Define normal subgroup of a group. Give one example.

Solution. Let G be a group ~~an~~ under given composition then a subgroup H of a group G is called normal if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.

Note. Since $xHx^{-1} = \{xhx^{-1} \mid h \in H, x \in G\}$

it means $xHx^{-1} \subseteq H$

Example. If $[G = \{1, -1, i, -i\}, \cdot]$ is a group

then $\{H = \{1, -1\}, \cdot\}$ is normal subgroup of G.

Q2. Define improper normal subgroup of a group G.

Ans. Let G be a group under given composition then two subgroups namely

$H_1 = \{e\}$, e is identity element of G.

and $H_2 = G$

are called improper normal subgroups of G.

Note. Rest normal subgroups are called proper normal subgroups of G.

Example. Let $[G = \{1, \omega, \omega^2\}, \cdot]$ is a group

Its ~~imp~~ improper normal subgroups are

$H_1 = (1)$ and $H_2 = G$.

Q 3. Define simple group. Give one example

Ans. A group G is called simple if it has only improper normal subgroups.

Example. $[G = \{1, \omega, \omega^2\}, \cdot]$ is simple as it has only ~~4~~ improper normal subgroups namely $H_1 = (1)$, $H_2 = G$.

Theorem 1. A subgroup H of a group G is normal iff $xHx^{-1} = H, \forall x \in G$.

Proof. Let H be a subgroup of a group G.

"If part" Suppose ~~H◁G~~ $xHx^{-1} = H$ —(1)

To prove ~~$xHx^{-1}=H$~~ $H \triangleleft G$

~~Proof ... ⇒ ...~~

~~⇒ ...~~ —(1)

given $xHx^{-1} = H \Rightarrow xHx^{-1} \subseteq H$

$\Rightarrow xhx^{-1} \in H$ $[\because xHx^{-1} \subseteq H]$

$\Rightarrow H \triangleleft G$

"only if part" Suppose $H \triangleleft G$

To prove $xHx^{-1} = H$

given $H \triangleleft G$

$\Rightarrow xhx^{-1} \in H$ $\{\because xHx^{-1} = \{xhx^{-1} \mid h \in H\}\}$

$\Rightarrow xHx^{-1} \subseteq H$ —(2)

$xHx^{-1} \subseteq H$

put $x = x^{-1}$

$\Rightarrow x^{-1}H(x^{-1})^{-1} \subseteq H$

$\Rightarrow x^{-1}Hx \subseteq H$

$\Rightarrow x(x^{-1}Hx)x^{-1} \subseteq xHx^{-1}$

$\Rightarrow xx^{-1}Hxx^{-1} \subseteq xHx^{-1}$

$\Rightarrow eHe \subseteq xHx^{-1}$ —(3)

$\Rightarrow H \subseteq xHx^{-1}$

$\therefore$ from (2) and (3)

$xHx^{-1} = H$

Theorem 2. A subgroup H of a group G is a normal subgroup of G if and only if each left coset of H in G is a right coset of H in G.

Proof. Let H be a subgroup of a group G

"If part" Suppose $H \triangleleft G$

To prove each left coset of H in G is a right coset of H in G

given $H \triangleleft G \Rightarrow xHx^{-1} = H$

$\Rightarrow xHx^{-1}x = Hx$ $[x^{-1}x = e]$

$\Rightarrow xHe = Hx$ $[He = H]$

$\Rightarrow xH = Hx$

"Only if part" Suppose each left coset of H in G is a right coset of H in G

To prove $H \triangleleft G$

Let $xH = Hy$ for some $x \in G, y \in G$. —(1)

Now let $x \in xH$ { $\because xH = \{xh \mid h \in H\}$ if we put $h = e$ we get $xe = x$ $\Rightarrow x \in xH$ }

$\Rightarrow x \in Hy$

$\Rightarrow Hx = Hy$ [$\because h \in H \Rightarrow Hh = H$] —(2)

$\therefore$ (1) and (2) $\Rightarrow xH = Hx$

$\Rightarrow xHx^{-1} = Hxx^{-1}$

$\Rightarrow xHx^{-1} = He$ [$\because xx^{-1} = e$]

$\Rightarrow xHx^{-1} = H$ [$\because He = H$]

$\Rightarrow H \triangleleft G$.

Theorem 3. A subgroup H of a group G is a normal subgroup of G if and only if the product of two right cosets of H is again a right coset of H in G.

Proof. Suppose H be a subgroup of a group G.

"~~If part~~" Let Ha and Hb be any two right cosets of H in G.

"If part" Suppose $H \triangleleft G$

To prove $(Ha)(Hb)$ is again a right coset of H in G

$(Ha)(Hb) = H(aH)b$ [Associativity]

$= H(Ha)b$ [$aH = Ha$]

$= HHab$ [Associativity]

$(Ha)(Hb) = Hab$ [$HH = H$]

$\Rightarrow Hab$ is a right coset of H in G as $(ab) \in G$.

"Only if part" Suppose the product of any two right cosets is again a right coset of H in G.

To prove $H \triangleleft G$

Let $x \in G \Rightarrow x^{-1} \in H$

then $(Hx)(Hx^{-1})$ is a right coset (by supposition)

$(Hx)(Hx^{-1}) = \{h_1 x h_2 x^{-1} \mid h_1 \in H, h_2 \in H\}$

Take $h_1 = h_2 = e$

$h_1 x h_2 x^{-1} = exex^{-1} = e \in HxHx^{-1} \Rightarrow$

but $e \in H$.

$\therefore$ $HxHx^{-1} = H$ [$Ha \cap Hb = \phi$ or $Ha = Hb$]

Now

Let $h_1 x h_2 x^{-1} \in HxHx^{-1}$

$\Rightarrow h_1 x h_2 x^{-1} \in H$ [$\because HxHx^{-1} = H$]

$\Rightarrow h_1^{-1}[h_1 x h_2 x^{-1}] \in h_1^{-1} H$

$\Rightarrow (h_1^{-1} h_1) x h_2 x^{-1} \in H$ [$h_1^{-1} H = H$, $h_1^{-1} \in H$]

$\Rightarrow e x h_2 x^{-1} \in H$

$\Rightarrow x h_2 x^{-1} \in H$ [$ex = x$]

$\Rightarrow H \trianglelefteq G$, proved.

Theorem 04: Prove that intersection of any two normal normal subgroups of a group G is a normal subgroup of the group G.

Proof. Let H_1 and H_2 be any two normal subgroups of a group G.

To prove $(H_1 \cap H_2) \trianglelefteq G$

Clearly $(H_1 \cap H_2)$ is a subgroup of G

because intersection of any two ~~normal~~ subgroups is a subgroup.

Let $h \in H_1 \cap H_2 \Rightarrow \underset{(i)}{h \in H_1}$ and $\underset{(ii)}{h \in H_2}$

Let $x \in G$

$\therefore$ (i) $\Rightarrow x \in G, h \in H_1 \Rightarrow x h x^{-1} \in H_1$ [$\because H \trianglelefteq G$]

(ii) $\Rightarrow x \in G, h \in H_2 \Rightarrow x h x^{-1} \in H_2$

$\therefore$ $x h x^{-1} \in H_1$, $x h x^{-1} \in H_2$

$\Rightarrow x h x^{-1} \in (H_1 \cap H_2)$

$\therefore$ $(H_1 \cap H_2) \trianglelefteq G$. proved.

Theorem 03. Let a be any element of a group G. Then two elements $x, y \in G$ give rise to the same conjugate of a iff they belong to the same right coset of the normalizer of a in G.

Hence show that if G is a finite group,

then $C_a = \dfrac{O(G)}{O[N(a)]}$, C_a = no of distinct elements in $C(a)$.

Proof. Let $N(a)$ be a normalizer of an element a of a ~~the~~ group G.

To prove $x, y \in G$ give rise to the same conjugate of a iff they belong to same right coset of $N(a)$.

We have $x, y \in G$ are in same right coset of $N(a)$

$\iff N(a)x = N(a)y$ $\quad [\because Ha = Hb \iff ab^{-1} \in H]$

$\iff xy^{-1} \in N(a)$ $\quad [N(a) = \{x \in G \mid xa = ax\}]$

$\iff a x y^{-1} = x y^{-1} a$

$\iff x^{-1} a x y^{-1} y = x^{-1} x y^{-1} a y$

$\iff x^{-1} a x e = e y^{-1} a y$ $\quad [\because y^{-1}y = e, x^{-1}x = e]$

$\iff x^{-1} a x = y^{-1} a y$

$\iff x$ and y give rise to same conjugate of a

First part is proved

Here we observe that

$x, y \in G$ are in same right coset of $N(a)$

$\iff x$ & y give rise to same conjugate of a

$\Rightarrow x, y \in G$ are in different right coset of $N(a)$

$\iff x$ & y give rise to different conjugate of a

Now $O[C(a)]$ = Total number of distinct elements of conjugate of a

= Total number of distinct ~~elements~~ right cosets of $N(a)$

= Index of $N(a)$ in G

$C_a = \dfrac{O(G)}{O(N(a))}$, proved

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Class equation of a group

We know that if G be a finite group $N(a)$ is a normalizer of an element a of G

then $o(G) = \dfrac{o(G)}{o[N(a)]}$

Let $a_1, a_2, a_3, \ldots, a_k \in G$, $k \le o(G)$ $\Rightarrow C(a_1), C(a_2), C(a_3) \ldots C(a_k)$ be conjugate classes of G

then we have $\Rightarrow G = C(a_1) \cup C(a_2) \cup \ldots \cup C(a_k)$

$$o(G) = \frac{o(G)}{o[N(a_1)]} + \frac{o(G)}{o(N(a_2))} + \cdots + \frac{o(G)}{o(N(a_k))}$$

$[\because o(G) = o[C(a_1)] + o[C(a_2)] + \cdots o[C(a_k)]]$

$$= \sum_{a = a_1, a_2, \ldots a_k} \frac{o(G)}{o[N(a)]}$$

$$o(G) = \sum_{a \in G} \frac{o(G)}{o[N(a)]}$$

which is called class equation of G

Note. $G = C(a_1) \cup C(a_2) \cup \cdots \cup C(a_k)$

$o(G) = o[C(a_1)] + o[C(a_2)] + \cdots + o[C(a_k)]$

$o(G) = \dfrac{o(G)}{o[N(a_1)]} + \dfrac{o(G)}{o[N(a_2)]} + \cdots + \dfrac{o(G)}{o[N(a_k)]}$

Self conjugate elements in a group G

An element a of a group G is called self conjugate if $a = x^{-1}ax$, $\forall x \in G$

Centre of a group G

By centre of a group G we mean the set of all self conjugate elements of G

Let Z denotes the centre of G then

$$Z = \{ z \in G \mid zx = xz, \forall x \in G \}$$

Note. Let $x \in Z \Rightarrow \exists a \in G$ such that $xa = ax, \forall a \in G$

$x \in Z \Rightarrow x \in N(a)$ $[\because N(a) = \{x \mid xa = ax\}$

$\Rightarrow Z \subseteq N(a).$

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Q. Prove that centre of a group G is normal subgroup of G.

Proof. Let Z be the centre of a group G then $Z = \{ z \in G \mid zx = xz,\ \forall x \in G \}$

Let $z_1 \in Z \Rightarrow z_1 x = x z_1$

$z_2 \in Z \Rightarrow z_2 x = x z_2$

To show $z_2^{-1} \in Z$

$$z_2^{-1}(z_2 x) z_2^{-1} = z_2^{-1}(x z_2) z_2^{-1}$$

$$\Rightarrow (z_2^{-1} z_2) x z_2^{-1} = z_2^{-1} x (z_2 z_2^{-1}) \quad \text{(Associativity)}$$

$$\Rightarrow e x z_2^{-1} = z_2^{-1} x e \quad [a.e = a]$$

$$\Rightarrow x z_2^{-1} = z_2^{-1} x$$

$$\Rightarrow z_2^{-1} \in Z$$

To show $z_1 z_2^{-1} \in Z$

$$x(z_1 z_2^{-1}) = (z_1 x) z_2^{-1} \quad [x z_1 = z_1 x]$$

$$= z_1 (x z_2^{-1}) \quad [\text{Associativity}]$$

$$= z_1 (z_2^{-1} x) \quad [x z_2^{-1} = z_2^{-1} x]$$

$$x(z_1 z_2^{-1}) = (z_1 z_2^{-1}) x \quad [\text{Associativity}]$$

$$\Rightarrow z_1 z_2^{-1} \in Z$$

$\Rightarrow Z$ is a subgroup of G

To show $Z \triangle G$

Let $x \in G$, $z \in Z$ then

$$x z x^{-1} = z x x^{-1} \quad [\because xz = zx]$$

$$= ze$$

$$= z \in Z$$

$$\Rightarrow x z x^{-1} \in Z$$

$\Rightarrow Z \triangle G$, proved.

Theorem 5. $a \in Z$ iff $N(a) = G$. If G is finite, $a \in G$

iff $o(N(a)) = o(G)$

Proof. Let Z be the centre of a group G.

Let $a \in Z \Rightarrow \exists\, x \in G$ — (A) such that $xa = ax$, $\forall x \in G$

Also $N(a) = \{x \in G \mid xa = ax\}$ — (B)

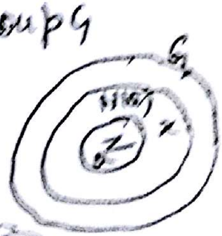
Now
$$a \in Z \Leftrightarrow ax = xa,\ \forall x \in G$$
$$\Leftrightarrow x \in N(a),\ \forall x \in G$$
$$\Leftrightarrow N(a) = G.$$

Since G is finite

$\therefore\ o[N(a)] = o[G]$.

Theorem 6. Let G be a finite group, Z be centre of G then the class eqn of G can be written as
$$o(G) = o(Z) + \sum_{a \notin Z} \frac{o(G)}{o[N(a)]}$$

Proof. Let Z be the centre of a finite group G



then we have
$$o(G) = \sum_{a \in Z} \frac{o(G)}{o[N(a)]} + \sum_{a \notin Z} \frac{o(G)}{o[N(a)]} \quad \text{— (1)}$$

Let $a \in Z \Rightarrow o[G] = o[N(a)]$
$$\Rightarrow \frac{o(G)}{o[N(a)]} = 1$$

i.e $C(a) = (a)$ [i.e conjugate class of a has only one element]

It means if $a_1, a_2, a_3, \dots a_k \in Z$ are $c(a_1) = (a_1), \dots c(a_k) = (a_k)$ $c(a_k) = (a_k)$

then $\cancel{c(a_1) = (a_1),\ c(a_2) = (a_2),\ \dots}$

i.e $\cancel{O(G = (a_1) \cup (a_2) \cup \dots \cup (a_k)}$

$\cancel{o(G) = 1 + 1 + 1 \dots + 1}$ (k terms)

i.e.
$$\sum_{a \in Z} \frac{o(G)}{o[N(a)]} = o(Z) \quad [\text{if } Z \text{ has } k \text{ distinct elements}]$$

$\therefore$ (1) $\Rightarrow$
$$o(G) = o(Z) + \sum_{a \notin Z} \frac{o(G)}{o[N(a)]}$$

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Theorem 07, If $o(G) = p^n$, p is prime number then prove that $Z \neq \{e\}$ [i.e. Z contains more than one elements]

Soln Let G be group such that $o(G) = p^n$

To prove $Z \neq \{e\}$ i.e. Z contains more than one elements [i.e. $o(z) > 1$]

We know that if Z be a centre of G then class equation of G is

$$o(G) = o(Z) + \sum_{a \notin Z} \frac{o(G)}{o(N(a))} \quad \text{--- (1)}$$

, $N(a)$ is a normalizer of a, $a \in G$.

$o(G) = p^n$, $o[N(a)]$ divides $o(G) \Rightarrow \exists$ n_a (positive integer) such that $1 \le n_a < n$

(1) Can be re-written as

$$o(Z) = o(G) - \sum_{a \notin Z} \frac{o(G)}{o(N(a))} \quad \text{--- (2)}$$

Now, $p \mid p^n$, p divides each of $\sum \frac{o(G)}{o(N(a))}$

$\Rightarrow$ p must divide $o(Z)$

$\Rightarrow$ $o(Z)$ must be ≥ 2

$\Rightarrow$ $o(Z) \neq \{e\}$. proved.

Corro. If $o(G) = p^2$, p is a prime then G is abelian

Proof. Let Z be the centre of the group G.

(i) if $o(Z) = o(G) \Rightarrow G$ is an abelian [$Z = \{z \in G \mid zx = xz, \forall x \in G\}$]

(ii) If $o(Z) \neq o(G)$
Let $o(Z) = p$ (suppose if possible)
$\Rightarrow \exists\, a \in G$ such that $a \notin Z$.
We know $Z \subseteq N(a)$
Now $a \in N(a)$, $Z \subseteq N(a)$ but $a \notin Z$
$\Rightarrow$ Elements in $N(a) > o(Z)$ $\Rightarrow o(N(a))$ must be equal to p^2
But $o(N(a))$ must divide $o(G)$
as $o(G)$ has three factors $1, p, p^2$
i $o(N(a)) = p^2 = o(G) \Rightarrow a \in Z$, Contradiction
ii $o(Z) = p^2 \Rightarrow G$ is abelian.